

Crystalline Ground States in Polyakov-loop extended Nambu–Jona-Lasinio Models

Jens Braun,^{1,2} Felix Karbstein,^{3,4} Stefan Rechenberger,¹ and Dietrich Roscher¹

¹*Institut für Kernphysik (Theoriezentrum), Technische Universität Darmstadt, D-64289 Darmstadt, Germany*

²*ExtreMe Matter Institute EMMI, GSI, Planckstraße 1, D-64291 Darmstadt, Germany*

³*Helmholtz-Institut Jena, Fröbelstieg 3, D-07743 Jena, Germany*

⁴*Theoretisch-Physikalisches Institut, Friedrich-Schiller-Universität Jena, Max-Wien-Platz 1, D-07743 Jena, Germany*

Nambu–Jona-Lasinio-type models have been used extensively to study the dynamics of the theory of the strong interaction at finite temperature and quark chemical potential on a phenomenological level. In addition to these studies, which are often performed under the assumption that the ground state of the theory is homogeneous, searches for the existence of crystalline phases associated with inhomogeneous ground states have attracted a lot of interest in recent years. In this work, we study the Polyakov-loop extended Nambu–Jona-Lasinio model and find that the existence of a crystalline phase is stable against a variation of the parametrization of the underlying Polyakov loop potential. To this end, we adopt two prominent parametrizations. Moreover, we observe that the existence of a quarkyonic phase depends crucially on the parametrization, in particular in the regime of the phase diagram where inhomogeneous chiral condensation is favored.

I. INTRODUCTION

The original motivation to search for inhomogeneous, i.e. crystalline, ground states in many-body systems goes back to the early ground-breaking works by *Fulde* and *Ferrell* as well as *Larkin* and *Ovchinnikov* who found that the ground state of superconducting materials as described within *Bardeen-Cooper-Schrieffer* (BCS) theory [1] is not necessarily homogeneous [2, 3]. In recent years, the possibility of having inhomogeneous ground states in a variety of different theories, such as the theory of the strong interaction, i.e. Quantum Chromodynamics (QCD), has attracted a lot of attention, see, e.g., Refs. [4–12].

The search for inhomogeneous phases in the phase diagram of QCD is inspired and paralleled by similar searches in related condensed-matter systems (see, e.g., Refs. [13–17]) and ultracold atomic gases (see, e.g., Refs. [18–20]) where the emergence of inhomogeneous condensates can be related to the in-medium formation of bosonic bound states with finite center of mass momentum [21, 22]. However, studies of – at least within the mean-field approximation – exactly solvable, relativistic field theories of strongly interacting fermions in low dimensions probably played the most important role in recent years [23–29]. The analyses of these low-dimensional models are most relevant to guide our understanding of fermionic theories in higher dimensions. Three-dimensional Nambu–Jona-Lasinio-type (NJL) models are close relatives of these low-dimensional models and are often employed as low-energy models for QCD. In particular, they are used to study the finite-temperature phase structure of QCD on a phenomenological level, see Refs. [30–32] for reviews. In recent years, studies of these models have been extended in various ways and the search for inhomogeneous phases has been put forward [6, 7]. In fact, depending on the actual choice for the parameters, an inhomogeneous phase has been found to exist at large quark chemical potential in this class of models [6, 7], see Ref. [9] for a recent review. This exotic

phase even persists to exist when the NJL model is set up with a nonlocal four-fermion interaction [33, 34].

In the past twenty years several extensions of the original NJL model [35, 36] have been introduced to bring it closer to QCD. Some focus has been laid on extensions which account at least partially for the missing dynamics of the gluon degrees of freedom. The so-called Polyakov-loop extended NJL (PNJL) model played a prominent role in this regard [37–39] and also inspired the discussion on the existence of a confined but chirally symmetric phase, the quarkyonic phase [40]. Within this extension, the order parameter for the deconfinement phase transition, the Polyakov loop, is coupled to the chiral quark dynamics as described by the NJL model and therefore it allows to study to some extent the effect of the confining gauge dynamics on the chiral phase structure of QCD. By now, first studies are available which aim at an understanding of the effect of the gauge dynamics on the chiral dynamics driving the formation of an inhomogeneous ground state at large quark chemical potential, see, e.g. [41, 42], and Ref. [9] for a review. Unfortunately, the parametrization of the Polyakov loop potential underlying PNJL models is ambiguous. In the original formulations of PNJL-type models, the backreaction of the quark dynamics on the Polyakov loop potential has not been taken into account, see Ref. [43] for a discussion of such effects. For example, the Polyakov loop potential in full QCD has an implicit dependence on the quark chemical potential and the number of quark flavors. The latter can readily be understood by recalling the dependence of the dynamically generated scale of QCD, namely Λ_{QCD} , on the number of quark flavors. This quantity directly affects the scaling of all (dimensionful) physical observables, such as the chiral phase transition temperature [44, 45]. This has also been studied in Ref. [46], where the dependence on the quark chemical potential has been considered as well, and used to amend Polyakov loop potentials entering QCD low-energy models. Taking this into account, it has been found that the chiral and deconfinement phase transition

lines remain close to each other up to large values of the quark chemical potential, implying the disappearance of the above-mentioned quarkyonic phase in these model studies [46].

Since the parametrization of the Polyakov loop potential affects the phase structure of these QCD models, particularly at large quark chemical potential [40, 46], it is also important to study the fate of inhomogeneous phases under a variation of this parametrization. In this paper, we address this question. In Sect. II, we introduce our model setup. To detect the onset of inhomogeneous phases, we employ the *fermion doubling trick* introduced in Ref. [47]. This approach has proven to reproduce the correct phase structure of the one-dimensional Gross-Neveu (GN) model in the mean-field approximation [25, 48, 49], including the existence of the inhomogeneous phase. Beyond the phenomenological questions being at the heart of this work, our present study also represents a further demonstration of the applicability of this trick [47]. In Sect. III, we present our results for the phase structure of the PNJL model and discuss the effect of the parametrization of the Polyakov loop potential. Our conclusions are given in Sect. IV.

II. MODEL

In the following we work in four-dimensional Euclidean spacetime. Focusing exclusively on the chiral limit, i.e. the limit of vanishing current quark masses, the action of the (local) PNJL model is given by [38, 39]

$$S = \int_{\tau} \int_{\vec{x}} \left\{ \mathcal{U}(\Phi, \bar{\Phi}, T) + \bar{\psi} (i\gamma_{\nu} (\partial_{\nu} - i\bar{g}A_0\delta_{\nu,0}) - i\mu\gamma_0) \psi + \frac{\bar{\lambda}}{2} \left[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\vec{t}\psi)^2 \right] \right\}, \quad (1)$$

where \mathcal{U} denotes the Polyakov loop potential, μ is the quark chemical potential, the t_i 's are the Pauli matrices, τ is the Euclidean time, $\int_{\tau} = \int_0^{\beta} d\tau$, and $\int_{\vec{x}} = \int d^3x$ accordingly, and finally $\beta = 1/T$ is the inverse temperature. The quark fields ψ interact via a local four-fermion interaction parametrized by the bare coupling $\bar{\lambda}$. Moreover, we have introduced a temporal gauge field A_0 which can be related to the order parameter for the deconfinement phase transition, namely the expectation value of the Polyakov loop $\langle\Phi(\vec{x})\rangle$. The corresponding variables $\Phi(\vec{x})$ and $\bar{\Phi}(\vec{x})$ read [50, 51]

$$\Phi(\vec{x}) = \frac{1}{N_c} \text{Tr}_c L(\vec{x}), \quad \bar{\Phi}(\vec{x}) = \frac{1}{N_c} \text{Tr}_c L^{\dagger}(\vec{x}), \quad (2)$$

with

$$L(\vec{x}) = \mathcal{P} \exp \left(i\bar{g} \int_0^{\beta} d\tau A_0(\tau, \vec{x}) \right). \quad (3)$$

Here, \mathcal{P} denotes path ordering. The color trace, Tr_c , is taken in the fundamental representation. The quantity \bar{g} is the bare gauge coupling, which can be conveniently

absorbed into a redefinition of the gauge field, and $N_c = 3$ is the number of colors.

The expectation value $\langle\Phi(\vec{x})\rangle$ measures whether center symmetry is realized in the ground state of the theory [52, 53]. A center-symmetric confining ground state is signaled by $\langle\Phi\rangle = 0$, whereas deconfinement, $\langle\Phi\rangle \neq 0$, is associated with center-symmetry breaking. Moreover, the negative logarithm of $\langle\Phi\rangle$ relates to the free energy of a static fundamental color source [52, 53]. In the presence of (light) dynamical quarks as in this work, the center symmetry is broken explicitly, i.e. $\langle\Phi\rangle \neq 0$ for all temperatures, rendering the deconfinement phase transition a crossover. Note that Φ and $\bar{\Phi}$ can be considered as independent fields. The expectation values of both are identical at $\mu = 0$ but differ at finite μ , $\langle\bar{\Phi}\rangle \geq \langle\Phi\rangle$ [54].

In our present study, the quark dynamics is coupled to the Polyakov loop variables Φ and $\bar{\Phi}$ via the temporal gauge field A_0 . Their dynamics is determined by the potential \mathcal{U} in the action (1). In pure gauge theory, the potential \mathcal{U} may be used to define an effective theory for the Polyakov loop dynamics in terms of the independent fields Φ and $\bar{\Phi}$. For \mathcal{U} , we choose the following ansatz which respects the $Z(3)$ center symmetry of the underlying $SU(3)$ gauge theory [55, 56]:

$$\frac{\mathcal{U}}{T^4} = -\frac{b_2}{2}\bar{\Phi}\Phi - \frac{b_3}{6}(\Phi^3 + \bar{\Phi}^3) + \frac{b_4}{4}(\bar{\Phi}\Phi)^2. \quad (4)$$

Restricting ourselves to constant and homogeneous Φ and $\bar{\Phi}$, and dropping fluctuations, the potential \mathcal{U} can be directly related to the associated effective action. Its minimization then yields the temperature-dependent expectation values $\langle\Phi\rangle$ and $\langle\bar{\Phi}\rangle$ defining the minimum of the potential as well as the thermodynamic quantities, such as the pressure $p(T) = -\mathcal{U}(\langle\Phi\rangle, \langle\bar{\Phi}\rangle, T)$. This is used to fit the parameters in Eq. (4) to data from lattice Monte Carlo simulations of $SU(3)$ Yang-Mills theory such that the behavior of thermodynamic quantities is consistent with these simulations and $\langle\Phi\rangle = \langle\bar{\Phi}\rangle$, as it should be. Here, we follow Refs. [39, 46] and employ

$$b_2(T) = a_0 + a_1 \left(\frac{T}{T_0} \right) + a_2 \left(\frac{T}{T_0} \right)^2 + a_3 \left(\frac{T}{T_0} \right)^3, \\ b_3 = 0.75, \quad b_4 = 7.5, \quad (5)$$

where

$$a_0 = 6.75, \quad a_1 = -1.95, \quad a_2 = 2.625, \quad a_3 = -7.44. \quad (6)$$

The value of T_0 may be assumed to depend on the number of quark flavors and the quark chemical potential. This then accounts for the fact that the dynamics of the gauge fields effectively generating the potential \mathcal{U} is in general affected by the quark dynamics. Following Ref. [46], we use $T_0 = 208 \text{ MeV}$ at $\mu = 0$ for two quark flavors. Moreover, we consider two different parametrizations which are distinguished by either choosing T_0 to be independent of μ or to be μ -dependent as put forward in Ref. [46]:

$$T_0(\mu) = T_{\tau} e^{-\frac{1}{\alpha_0 b(\mu)}}, \quad (7)$$

where, for two massless quark flavors,

$$b(\mu) = \frac{29}{6\pi} - \frac{32}{\pi} \left(\frac{\mu}{T_\tau} \right)^2, \quad (8)$$

and

$$\alpha_0 = 0.304, \quad T_\tau = 1770 \text{ MeV}. \quad (9)$$

These parameters basically reflect the fact that the gauge coupling is assumed to be fixed at the τ -mass scale $m_\tau \approx T_\tau$, $\alpha_0 \approx \alpha(m_\tau)$.

Let us now turn to the computation of the effective potential from the action (1). To this end, we resort to the mean-field approximation and perform the conventional Hubbard-Stratonovich transformation of the action (1) which introduces auxiliary bosonic fields into the action (see, e.g., Refs. [30, 31, 57]),

$$\sigma \sim (\bar{\psi}\psi), \quad \vec{\pi} \sim (\bar{\psi}i\gamma_5\vec{t}\psi). \quad (10)$$

From a phenomenological point of view, these fields may be associated with the σ and pion fields, respectively. In our search for inhomogeneous ground states, we restrict ourselves to one of the most simple inhomogeneous

ansätze for the order parameter fields,

$$\sigma(\vec{x}) = \bar{\sigma} \cos(2\vec{Q} \cdot \vec{x}), \quad \vec{\pi}(\vec{x}) = 0, \quad (11)$$

with amplitude $\bar{\sigma}$ and wave-vector \vec{Q} . Another simple and prominent ansatz is the complex chiral density wave, cf., e.g. [58, 59]. For the latter, the (mean field) effective potential may even be computed exactly. However, it comes along with a generally nonvanishing parity-breaking order parameter $\langle \vec{\pi}(\vec{x}) \rangle$.

For $Q \equiv |\vec{Q}| = 0$, Eq. (11) reduces to the ansatz conventionally used in studies where the possibility of inhomogeneous chiral condensation is not taken into account. Switching to momentum space and employing the fermion doubling trick [47], we assume that the quark fields $\psi_n(\vec{p} - \vec{Q})$ and $\psi_n(\vec{p} + \vec{Q})$ are independent degrees of freedom in the Hubbard-Stratonovich-transformed action and integrate them out straightforwardly, see also Sect. III for a discussion of the limitations coming along with this assumption.

Choosing the temporal gauge field A_0 to be in the Cartan subalgebra and time- as well as space-independent [37–39], we obtain the following result for the effective potential Ω :

$$\begin{aligned} \Omega(\Phi, \bar{\Phi}, \bar{\sigma}, T, \mu, Q) = & \mathcal{U}(\Phi, \bar{\Phi}, T) + \frac{T}{V} \int_{\vec{\tau}} \int_{\vec{x}} \frac{\sigma(\vec{x})^2}{2\lambda} - 2T \int_{\vec{p}} \sum_{\pm} \left\{ \ln \left[1 + 3 \left(\Phi + \bar{\Phi} e^{(\mu - E_{\pm})/T} \right) e^{(\mu - E_{\pm})/T} + e^{3(\mu - E_{\pm})/T} \right] \right. \\ & \left. + \ln \left[1 + 3 \left(\bar{\Phi} + \Phi e^{-(\mu + E_{\pm})/T} \right) e^{-(\mu + E_{\pm})/T} + e^{3(\mu + E_{\pm})/T} \right] \right\} - 6 \int_{\vec{p}} \{E_+ + E_-\} \end{aligned} \quad (12)$$

with V being the spatial volume and

$$E_{\pm} = \sqrt{\vec{p}^2 + \vec{Q}^2 + \bar{\sigma}^2 \pm 2\sqrt{(\vec{p} \cdot \vec{Q})^2 + \bar{\sigma}^2 \vec{Q}^2}}. \quad (13)$$

Moreover, we choose a sharp ultraviolet (UV) cutoff Λ to regularize the space-like momentum integrals, i.e. $\int_{\vec{p}} := \int_{|\vec{p}| < \Lambda} \frac{d^3 p}{(2\pi)^3}$. Note that we regularize both the zero- and finite-temperature contributions in Eq. (12).¹

Similar to conventional (homogeneous) PNJL model studies [39, 46], the ground state for a given temperature and chemical potential is then determined by searching numerically for the saddle point in the multidimensional space spanned by the quantities Φ , $\bar{\Phi}$, $\bar{\sigma}$, and Q . The coupling constant $\bar{\lambda}$ and the UV cutoff Λ are tuned to obtain

the constituent quark mass $m_q \approx \bar{\sigma}_0 = 325 \text{ MeV}$ and the pion decay constant $f_\pi = 92.4 \text{ MeV}$ independently of Q at $T = \mu = 0$. Here, $\bar{\sigma}_0$ denotes the value of $\bar{\sigma}$ minimizing the effective potential. To be specific, in accordance with Ref. [39], this is achieved by choosing $\bar{\lambda}\Lambda^2 \approx 4.34$ and $\Lambda = 651 \text{ MeV}$.

III. PHASE STRUCTURE

A. Chiral Phase Structure

Let us now discuss the phase structure of the PNJL model as obtained from the two different parameterizations of the Polyakov loop potential in terms of the quantity T_0 , i.e. $T_0 = \text{const.}$ and the μ -dependent ansatz (7) for T_0 , respectively.

In Fig. 1 we show our result for the chiral phase diagram of the PNJL model for the parametrization (7) in the plane spanned by the temperature measured in units of the chiral phase transition temperature at $\mu = 0$, $T_{\chi,0} \equiv T_\chi(\mu = 0)$, and the quark chemical potential μ in units of the constituent quark mass m_q . For $\mu = 0$, we find $T_\chi = 227 \text{ MeV}$. Note that, as discussed in the previ-

¹ In principle, it is possible to regularize only the zero-temperature contributions as the finite-temperature contributions are finite anyways. We have checked both types of regularization and found that, e.g., the chiral phase transition temperature at $\mu = 0$ increases by about 20 MeV when both the zero- and finite-temperature contributions are regularized consistently with our sharp-cutoff prescription.

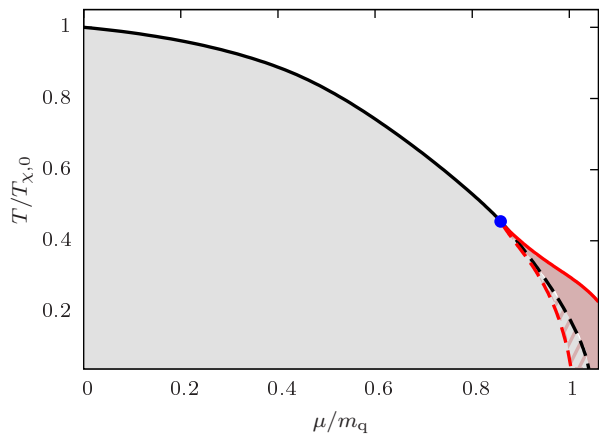


Figure 1. (color online) Chiral phase diagram of the PNJL model with two massless quark flavors in the plane spanned by the temperature in units of $T_{\chi,0} \equiv T_{\chi}(\mu = 0)$ and the quark chemical potential in units of m_q obtained with a μ -dependent T_0 -parameter. The black solid/dashed line denotes the 2nd/1st-order transition for the homogeneous setup, i.e., setting $Q \equiv 0$ in Eq. (11). Both lines meet at a critical point (blue dot). The energetically favored inhomogeneous chiral condensate is found between the red solid and red dashed line, being of 2nd and 1st order respectively.

ous section, this value depends on the regularization procedure for a given set of parameters in the presently used mean-field approximation. Since we are primarily interested in a qualitative discussion of the effect of the gauge dynamics as parametrized in terms of the Polyakov loop potential, we do not tune this value any further here although it is somewhat larger than naively expected for QCD with two massless quark flavors.

The chiral phase structure as obtained by employing a constant T_0 agrees qualitatively with the one shown in Fig. 1 which has been computed by using the ansatz (7) for T_0 . However, there are differences in the behavior of the Polyakov loop in particular at large values of the chemical potential to be analyzed below.

Let us start by discussing how the chiral phase structure is altered when we allow for the existence of inhomogeneous phases. Not treating Q in Eq. (12) as a variational parameter in our calculation but setting it to zero, we recover the conventional homogeneous chiral phase structure of the PNJL model, see, e.g., Ref. [32] for a review. To be more specific, in Fig. 1, the gray-shaded area bounded by the black solid (second-order chiral phase transition) and black dashed (first-order chiral phase transition) line is associated with the spontaneous formation of a homogeneous chiral condensate. The white area depicts the chirally symmetric phase.

Considering now Q as a variational parameter, we observe the emergence of an inhomogeneous phase associated with a finite spatially varying order parameter at large quark chemical potential. This regime is bounded

by the red dashed (first-order transition²) and red solid (second-order chiral transition) line. Here, we have $\bar{\sigma} > 0$ and $Q > 0$ in the ground state of the effective potential (12). Notably, the first-order transition line is shifted to smaller values of the chemical potential as compared to the $Q = 0$ case, implying that the size of the regime governed by a homogeneous chiral condensate (gray-shaded area) shrinks. Note that we only show the phase diagram in the regime $\mu/m_q \lesssim 1.1$. For larger values of μ , the results suffer strongly by the presence of the finite UV cutoff Λ which also constrains the domain of validity of the PNJL model as a low-energy model for QCD, independent of the details of the chosen regularization scheme (e.g. sharp cutoff or exponential cutoff). In any case, the general features of the chiral phase structure observed in our present work are in accordance with previous studies, see Ref. [9] for a review.

Before we analyze the effect of the two different parametrizations of the Polyakov loop potential, let us discuss the strengths and the limitations of our present approach. First of all, we work here in the mean-field approximation. Thus, fluctuation effects are not taken into account but they are certainly relevant to clarify conclusively whether an inhomogeneous phase is energetically favored at large chemical potential. To address this issue, it may in the future be worthwhile to study, e.g., the renormalization group (RG) flow of the auxiliary-field propagators as recently done in the context of ultracold Fermi gases to detect the onset of inhomogeneous condensation [22]. Second, our search for inhomogeneous phases is based on the fermion doubling trick [47] which allows to detect exactly the position of general second-order chiral phase transitions but only allows for an approximate determination of transition lines between phases with a homogeneous and an inhomogeneous chiral condensate (red dashed line in Fig. 1). In the context of the one-dimensional GN model, however, it has been shown numerically and argued analytically that at least the location of this type of phase transition line is also reproduced reasonably well.

Finally, we have restricted ourselves to a simple single-cosine ansatz (11) for the inhomogeneous ground state in the σ channel. Again, for the GN model in one dimension, it has been shown that the exact ground-state solution approaches this single-cosine form close the second-order phase boundary [25, 48, 49]. Although the predictive power of the ansatz (11) has been demonstrated for the one-dimensional GN model [47], and also for one-dimensional non-relativistic field theories [21], the exact functional form of the energetically most favorable inhomogeneity in higher dimensional GN- or NJL-type mod-

² Note that this first-order transition between the phase with a homogeneous chiral condensate and the inhomogeneous phase is not related to chiral symmetry breaking but to translation symmetry breaking as measured by the value of Q in the ground state.

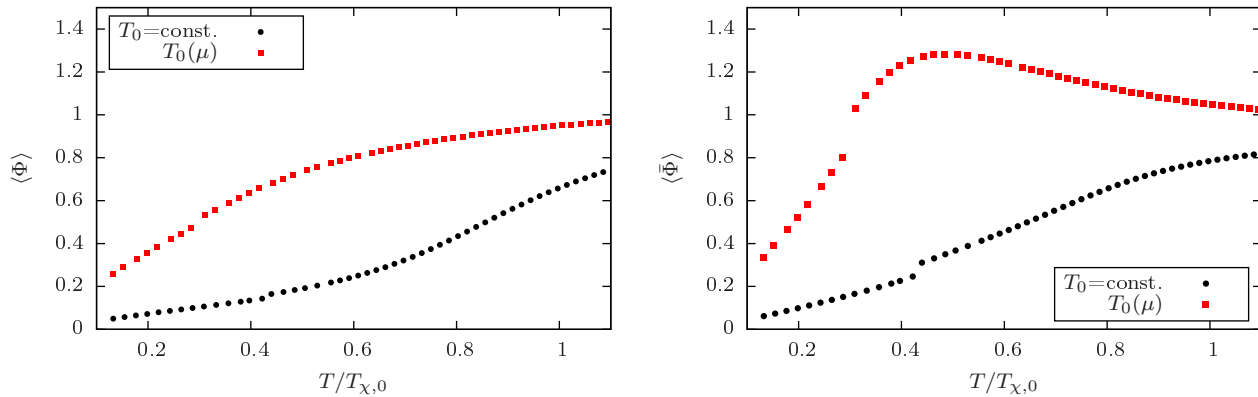


Figure 2. (color online) Polyakov loop (left panel) and adjoint Polyakov loop (right panel) as a function of the temperature in units of $T_{\chi,0}$ for the specific choice of $\mu/m_q = 1.02$ for both $T_0 = 208$ MeV and a μ -dependent parametrization of T_0 , see main text for details.

els is still unknown. In fact, there is a plethora of possible functional forms for the inhomogeneous ground state beyond our present one-dimensional ansatz (11) in the σ channel, ranging from so-called chiral spirals [7] to higher dimensional inhomogeneities [9].

In this respect, we add that, in our study with μ -dependent T_0 , we find a very thin spike of the inhomogeneous phase (width $\ll 1$ MeV in T -direction) which ‘cuts’ into the phase with a homogeneous chiral condensate starting from the so-called *Lifshitz* point (blue dot in Fig. 1). This may be considered as an indication that our ansatz for the inhomogeneity is not sufficient to determine correctly the location of the first-order transition line (red dashed line). On the other hand, it could very well be an artifact generated by the specific parametrization used here for the Polyakov loop potential or even a numerical artifact. Indeed, we have not observed indications for this in our studies with constant T_0 as well as in our benchmark studies of the conventional NJL model.

In any case, based on the ansatz (11), our present approach allows to detect regimes of inhomogeneous chiral condensation in the phase diagram in a numerically inexpensive way as it only relies on the evaluation of the effective potential Ω , as also usually done in conventional PNJL model studies only allowing for homogeneous chiral condensation. In particular, our approach allows to study efficiently the interplay of inhomogeneous condensation and confining dynamics as modelled by the Polyakov loop potential and can therefore help to guide numerically more costly approaches, such as exact diagonalization methods.

B. Effect of Confining Dynamics

Let us now discuss the effect of the parametrization of the Polyakov loop potential on the phase structure, with an emphasis on the regime at large chemical potential. Using a μ -independent constant value for T_0 , it has

been found that a quarkyonic phase emerges where the ground state is confined but chirally symmetric [40]. On the other hand, it has been observed that this phase diminishes or even vanishes completely when T_0 is assumed to be μ -dependent as given in Eq. (7), see Ref. [46]. In our search for inhomogeneous phases, we have used these two prominent and frequently used choices for T_0 . As already mentioned above, we observe that the chiral phase structure does not depend on our choice for T_0 on a qualitative level. Quantitatively, we find that the *Lifshitz* point is shifted to lower temperatures and larger values of the quark chemical potential when we choose T_0 to be μ -dependent. As a direct consequence, we observe that the inhomogeneous phase tends to shrink in this case.

We now turn to the deconfinement order parameter. Whereas our results for the deconfinement and the chiral phase transition line are in accordance with previous results in the regime to the left of the critical point [32, 40, 46], they differ to the right of this point. Setting $T_0 = \text{const.}$, we find that, for a given fixed value of μ , the deconfinement phase transition as measured in terms of $\langle \Phi \rangle$ (and $\langle \bar{\Phi} \rangle$) occurs at temperatures well above the chiral phase transition separating the inhomogeneous phase and the chirally symmetric phase, see Fig. 2. This suggests the existence of a confined phase with restored chiral symmetry [9], i.e. a quarkyonic phase. A comment is in order here. Since we work with light dynamical quarks, the deconfinement transition is a crossover rather than a true phase transition. Thus, the definition of the actual phase transition temperature is to some extent ambiguous. For example, we may use $\langle \Phi \rangle(T_d) = 0.5$ to define the deconfinement temperature T_d . In any case, we have only $\langle \Phi \rangle > 0.5$ and $\langle \bar{\Phi} \rangle > 0.5$ for $T/T_{\chi,0} \gtrsim 0.8$ and $T/T_{\chi,0} \gtrsim 0.7$, respectively. However, we observe a sudden steep³ increase in the results for $\langle \Phi \rangle$ and $\langle \bar{\Phi} \rangle$ as a

³ Within our numerical precision, we observe that the change in $\langle \Phi \rangle$ and $\langle \bar{\Phi} \rangle$ is very steep at this point but still continuous.

function of temperature in this part of the phase diagram, e.g. at $T/T_{\chi,0} \approx 0.43$ for $\mu/m_q = 1.02$ as illustrated by the black circles in Fig. 2. In the low-temperature regime, $\langle\Phi\rangle$ and $\langle\bar{\Phi}\rangle$ are still small, suggesting that the inhomogeneous phase is well in the effectively confined phase. The steep increase can be traced back to the fact that the system undergoes a second-order chiral phase transition from the inhomogeneous phase to the chirally symmetric phase at this point. Note that we only depict the phase diagram for the μ -dependent parametrization in Fig. 1. We expect that this increase is smeared out in the case of finite current quark masses when the chiral phase transition turns into a crossover as well.

The situation is different when we choose T_0 to depend on μ . We then find that the deconfinement phase transition temperature is shifted to lower temperatures. This is illustrated in Fig. 2 where we show $\langle\Phi\rangle$ and $\langle\bar{\Phi}\rangle$ (red squares) as a function of $T/T_{\chi,0}$ for $\mu/m_q = 1.02$. Both $\langle\Phi\rangle$ and $\langle\bar{\Phi}\rangle$ now increase more rapidly as a function of temperature. Again, we observe a steep increase in these quantities at low temperatures which is driven by the chiral transition from the inhomogeneous to the chirally symmetric phase at $T = T_\chi(\mu)$. In fact, we find that $\langle\Phi\rangle \approx 0.5$ at $T = T_\chi(\mu)$, suggesting that the chiral phase transition temperature coincides approximately with the deconfinement phase transition temperature T_d . The adjoint Polyakov loop $\langle\bar{\Phi}\rangle$ increases even stronger than $\langle\Phi\rangle$ at low temperatures. In summary, when choosing T_0 to be μ -dependent, we do not observe the emergence of a quarkyonic phase as found for $T_0 = \text{const.}$, even in the regime where inhomogeneous chiral condensation is found to be energetically favored.

IV. CONCLUSIONS

In the present work we have discussed the phase structure of the two-flavor PNJL model in the chiral limit as an effective low-energy model for QCD. We have particularly focussed on a discussion of the existence of inhomogeneous phases at low temperatures and large values of the chemical potential and on the question how these phases are affected by the confining dynamics being parametrized in terms of the Polyakov loop potential. To this end, we have computed the effective potential with the fermion doubling trick [47] and searched numerically for inhomogeneous ground states. We parametrized the latter by using a simple single-cosine ansatz in the σ channel with the amplitude and the frequency as variational parameters in the numerical search for the ground state. This choice for the inhomogeneity is motivated by the exact solution of the one-dimensional GN model where the inhomogeneity approaches this functional form close to the phase boundary [25, 48, 49]. For higher dimensional NJL- and GN-type models, even the general functional form of the exact ground state is unknown. In

this sense, we only probe one specific form of the inhomogeneity. For our present study, however, we consider this to be sufficient as we primarily aimed at a qualitative understanding of how the emergence of inhomogeneous phases is affected by the confining gauge dynamics. To this end, we expect that the knowledge of the exact functional form of the inhomogeneity is not crucial. The strength of our present approach is that it allows to study different parametrizations and functional forms of the Polyakov loop potentials at comparatively low computational cost. Here, we have used two prominent and frequently used parametrizations of this potential distinguished by a μ -independent and a μ -dependent choice for the parameter T_0 , respectively.

In accordance with earlier studies [40, 46], where the possibility of inhomogeneous chiral condensation has not been taken into account explicitly, we observe for constant T_0 that a quarkyonic phase emerges between a low-temperature confined phase with broken chiral symmetry and a high-temperature deconfined chirally symmetric phase, even when we now allow explicitly for inhomogeneous condensation in our studies, in accordance with Ref. [9]. Taking into account a possible μ -dependence of T_0 , we observe that the quarkyonic phase vanishes and we are only left with a low-temperature confined phase with spontaneously broken chiral symmetry and a deconfined chirally symmetric phase, even in the regime of the phase diagram where inhomogeneous condensation is energetically favored. Moreover, we find that the size of the inhomogeneous phase shrinks when we employ a μ -dependent T_0 .

Clearly, our present work only demonstrates that the dynamics at large chemical potential is significantly affected by the parametrization of the gauge dynamics in terms of the Polyakov loop potential. The present study is not completely conclusive with respect to the true phase structure of PNJL-type models (let alone QCD) at large chemical potential. It only points to presently still existing uncertainties in our understanding of even the general phase structure of QCD in this regime. Further studies with different functional forms of the Polyakov loop potential may be required to improve our understanding in this respect. In particular, improved computations of this potential based on, e.g., functional RG approaches along the lines of Refs. [43, 60, 61] taking into account the backreaction of the quarks on the gluodynamics, may be necessary to further narrow down these uncertainties and ultimately help to better the predictions for the equation of state at high densities.

Acknowledgments.— The authors would like to thank J. M. Pawłowski for useful discussions. JB, SR, and DR acknowledge support by HIC for FAIR within the LOEWE program of the State of Hesse. In addition, JB and SR acknowledge support by the Deutsche Forschungsgemeinschaft (DFG) through contract SFB 634. JB and DR acknowledge support by the DFG under Grant BR 4005/2-1.

-
- [1] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, *Phys. Rev.* **108**, 1175 (1957).
- [2] P. Fulde and R. A. Ferrell, *Phys. Rev.* **135**, A550 (1964).
- [3] A. I. Larkin and Y. N. Ovchinnikov, *Sov. Phys. JETP* **20**, 762 (1965).
- [4] B. Bringoltz, *JHEP* **0703**, 016 (2007), arXiv:hep-lat/0612010 [hep-lat].
- [5] B. Bringoltz, *Phys. Rev.* **D79**, 125006 (2009), arXiv:0901.4035 [hep-lat].
- [6] D. Nickel, *Phys. Rev.* **D80**, 074025 (2009), arXiv:0906.5295 [hep-ph].
- [7] T. Kojo, Y. Hidaka, L. McLerran, and R. D. Pisarski, *Nucl. Phys.* **A843**, 37 (2010), arXiv:0912.3800 [hep-ph].
- [8] D. Mueller, M. Buballa, and J. Wambach, *Eur. Phys. J.* **A49**, 96 (2013), arXiv:1303.2693 [hep-ph].
- [9] M. Buballa and S. Carignano, *Prog. Part. Nucl. Phys.* **81**, 39 (2015), arXiv:1406.1367 [hep-ph].
- [10] S. Carignano, M. Buballa, and B.-J. Schaefer, *Phys. Rev.* **D90**, 014033 (2014), arXiv:1404.0057 [hep-ph].
- [11] T.-G. Lee, E. Nakano, Y. Tsue, T. Tatsumi, and B. Friman, *Phys. Rev.* **D92**, 034024 (2015), arXiv:1504.03185 [hep-ph].
- [12] A. Heinz, F. Giacosa, M. Wagner, and D. H. Rischke, (2015), arXiv:1508.06057 [hep-ph].
- [13] R. Jackiw and J. Schrieffer, *Nucl. Phys.* **B190**, 253 (1981).
- [14] J. Mertsching and H. J. Fischbeck, *phys. stat. sol. (b)* **103**, 783 (1981).
- [15] K. Machida and H. Nakanishi, *Phys. Rev.* **B 30**, 122 (1984).
- [16] A. Chodos, H. Minakata, and F. Cooper, *Phys. Lett.* **B449**, 260 (1999), arXiv:hep-ph/9812305 [hep-ph].
- [17] H. Kleinert and E. Babaev, *Phys. Lett.* **B438**, 311 (1998), arXiv:hep-th/9809112 [hep-th].
- [18] A. Bulgac and M. M. Forbes, *Phys. Rev. Lett.* **101**, 215301 (2008), arXiv:0804.3364 [cond-mat.supr-con].
- [19] J. E. Baarsma and H. T. C. Stoof, arXiv:1212.5450 [cond-mat.quant-gas].
- [20] L. Radzihovsky, *Physica C: Superconductivity* **481**, 189 (2012).
- [21] D. Roscher, J. Braun, and J. E. Drut, *Phys. Rev.* **A89**, 063609 (2014), arXiv:1311.0179 [cond-mat.quant-gas].
- [22] D. Roscher, J. Braun, and J. E. Drut, *Phys. Rev.* **A91**, 053611 (2015), arXiv:1501.05544 [cond-mat.quant-gas].
- [23] V. Schon and M. Thies, hep-th/0008175 [hep-th].
- [24] M. Thies, *Phys. Rev.* **D69**, 067703 (2004), arXiv:hep-th/0308164 [hep-th].
- [25] O. Schnetz, M. Thies, and K. Urlichs, *Annals Phys.* **314**, 425 (2004), arXiv:hep-th/0402014 [hep-th].
- [26] O. Schnetz, M. Thies, and K. Urlichs, *Annals Phys.* **321**, 2604 (2006), arXiv:hep-th/0511206 [hep-th].
- [27] G. Basar and G. V. Dunne, *Phys. Rev. Lett.* **100**, 200404 (2008), arXiv:0803.1501 [hep-th].
- [28] G. Basar and G. V. Dunne, *Phys. Rev.* **D78**, 065022 (2008), arXiv:0806.2659 [hep-th].
- [29] G. Basar, G. V. Dunne, and M. Thies, *Phys. Rev.* **D79**, 105012 (2009), arXiv:0903.1868 [hep-th].
- [30] S. P. Klevansky, *Rev. Mod. Phys.* **64**, 649 (1992).
- [31] M. Buballa, *Phys. Rept.* **407**, 205 (2005), arXiv:hep-ph/0402234 [hep-ph].
- [32] K. Fukushima, *J. Phys. G* **G39**, 013101 (2012), arXiv:1108.2939 [hep-ph].
- [33] J. P. Carlomagno, D. G. Dumm, and N. N. Scoccola, *Phys. Lett.* **B745**, 1 (2015), arXiv:1411.0909 [hep-ph].
- [34] J. P. Carlomagno, D. G. Dumm, and N. N. Scoccola, (2015), arXiv:1507.01560 [hep-ph].
- [35] Y. Nambu and G. Jona-Lasinio, *Phys. Rev.* **122**, 345 (1961).
- [36] Y. Nambu and G. Jona-Lasinio, *Phys. Rev.* **124**, 246 (1961).
- [37] P. N. Meisinger and M. C. Ogilvie, *Phys. Lett.* **B379**, 163 (1996), arXiv:hep-lat/9512011 [hep-lat].
- [38] K. Fukushima, *Phys. Lett.* **B591**, 277 (2004), hep-ph/0310121.
- [39] C. Ratti, M. A. Thaler, and W. Weise, *Phys. Rev.* **D73**, 014019 (2006), arXiv:hep-ph/0506234.
- [40] L. McLerran and R. D. Pisarski, *Nucl. Phys.* **A796**, 83 (2007), arXiv:0706.2191 [hep-ph].
- [41] T. L. Partyka and M. Sadzikowski, *Acta Phys. Polon.* **B42**, 1305 (2011), arXiv:1011.0921 [hep-ph].
- [42] S. Carignano, D. Nickel, and M. Buballa, *Phys. Rev.* **D82**, 054009 (2010), arXiv:1007.1397 [hep-ph].
- [43] L. M. Haas, R. Stiele, J. Braun, J. M. Pawłowski, and J. Schaffner-Bielich, *Phys. Rev.* **D87**, 076004 (2013), arXiv:1302.1993 [hep-ph].
- [44] J. Braun and H. Gies, *Phys. Lett.* **B645**, 53 (2007), arXiv:hep-ph/0512085 [hep-ph].
- [45] J. Braun and H. Gies, *JHEP* **06**, 024 (2006), arXiv:hep-ph/0602226 [hep-ph].
- [46] B.-J. Schaefer, J. M. Pawłowski, and J. Wambach, *Phys. Rev.* **D76**, 074023 (2007), arXiv:0704.3234 [hep-ph].
- [47] J. Braun, S. Finkbeiner, F. Karbstein, and D. Roscher, *Phys. Rev.* **D91**, 116006 (2015), arXiv:1410.8181 [hep-ph].
- [48] M. Thies and K. Urlichs, *Phys. Rev.* **D67**, 125015 (2003), arXiv:hep-th/0302092 [hep-th].
- [49] M. Thies, *J. Phys.* **A39**, 12707 (2006), arXiv:hep-th/0601049 [hep-th].
- [50] A. M. Polyakov, *Phys. Lett.* **B72**, 477 (1978).
- [51] L. Susskind, *Phys. Rev.* **D20**, 2610 (1979).
- [52] B. Svetitsky, *Phys. Rept.* **132**, 1 (1986).
- [53] J. Greensite, *Prog. Part. Nucl. Phys.* **51**, 1 (2003), arXiv:hep-lat/0301023 [hep-lat].
- [54] A. Dumitru, R. D. Pisarski, and D. Zschesche, *Phys. Rev.* **D72**, 065008 (2005), arXiv:hep-ph/0505256 [hep-ph].
- [55] R. D. Pisarski, *Phys. Rev.* **D62**, 111501 (2000), arXiv:hep-ph/0006205 [hep-ph].
- [56] R. D. Pisarski, *Phys. Rev.* **D74**, 121703 (2006), arXiv:hep-ph/0608242 [hep-ph].
- [57] J. Braun, *J. Phys.* **G39**, 033001 (2012), arXiv:1108.4449 [hep-ph].
- [58] M. Sadzikowski and W. Broniowski, *Phys. Lett.* **B488**, 63 (2000), arXiv:hep-ph/0003282 [hep-ph].
- [59] E. Nakano and T. Tatsumi, *Phys. Rev.* **D71**, 114006 (2005), arXiv:hep-ph/0411350 [hep-ph].
- [60] J. Braun, H. Gies, and J. M. Pawłowski, *Phys. Lett.* **B684**, 262 (2010), arXiv:0708.2413 [hep-th].
- [61] J. Braun, L. M. Haas, F. Marhauser, and J. M. Pawłowski, *Phys. Rev. Lett.* **106**, 022002 (2011), arXiv:0908.0008 [hep-ph].